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THE CONSTRUCTION OF SYSTEMS WITH ASYMPTOTICALLY STABLE PROGRAMMED CONSTRAINTS†

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The method of constructing a family of Lyapunov functions to investigate the stability "in the small" of a perturbed motion, specified in the form of a law of motion of a mechanical system [1], which has also been used to construct generalized systems possessing an asymptotically stable programmed motion [2], is extended to generalized systems possessing asymptotically stable programmed constraints. Examples of the use of this procedure in the problem of stabilizing the programmed manifold of a manipulator on a moving base and to stabilize the programmed orientation of a pursuing body are presented. © 2002 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Suppose the motions of a system are described by the equation

$$A_0(\dot{x},x,t)\ddot{x} = B_0(\dot{x},x,t) + M_0(\dot{x},x,t)u \tag{1.1}$$

where A_0 and M_0 are $(n \times n)$ and $(n \times r)$ matrices, x and B_0 are n-dimensional vectors and u is an r-dimensional vector.

System (1.1) is more general compared with mechanical systems since, unlike them, the matrix A_0 may depend on $\dot{x}$, cannot be symmetric and positive-definite but must be non-singular.

The problem entails finding expressions for the vector $u(x, x, t)$ for which system (1.1) possesses programmed constraints

$$\omega_1(x,t) = 0, \quad \omega_2(\dot{x},x,t) = 0 \tag{1.2}$$

which are asymptotically stable "in the large" where ω_1, ω_2 are specified k- and m-dimensional vector functions and $k + m \leqslant r$. It is assumed that the vectors ω_1, ω_2, B_0 and the elements of the matrices A_0, M_0 are bounded and continuously differentiable functions in a certain bounded domain $G(x, \dot{x})$ when $t \geqslant t_0$ which includes the manifold (1.2) and a certain neighbourhood of it.

Moreover, it is assumed that the determinants of the matrices $M_0M_0^T$, $(\partial\omega_1/\partial x)(\partial\omega_1/\partial x)^T$, $(\partial\omega_2/\partial\dot{x})(\partial\omega_2/\partial\dot{x})^T$, which are Gram determinants, do not vanish in the domain G.

Remark. The concept of a programmed constraint was introduced by the author in his doctoral dissertation (1972) and consists of the fact that a programme of motion can be specified as a law of motion or, in a more general form, as the manifold (1.2), which is analogous to the equations of the constraints imposed on the constrained mechanical systems. Unlike constrained mechanical systems, the phase states of the controlled system may or may not satisfy Eqs (1.2) since the controlled system, generally speaking, is a free system while conditions (1.2) are solely indicative of the fact that the manifold (1.2) must be integral for the equations of motion of the system. In order to achieve this aim, the active control forces acting on the system are chosen so that, when the initial states of the system satisfy (1.2), the system constructed behaves in exactly the same way as the constrained system on which constraints of the form of (1.2) have been imposed. The basic property of controlled systems and their similarity with constrained mechanical systems is contained in this.
Note that a programme, which is specified in the form of a law of motion, is a special case of (1.2) when $k = n$.

Below, the quantities ω_1 and ω_2 will be taken as measures of the deviation of the motions (1.1) from the manifold (1.2).

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2. CONSTRUCTION OF A SYSTEM WITH CONTINUOUS CONTROL

By virtue of (1.1), on differentiating the first equation of (1.2) twice with respect to time and the second equation of (1.2) once, we obtain

$$\ddot{\omega}_1 = B_1(\dot{x},x,t)+\left(\frac{\partial\omega_1}{\partial x}\right)^{\mathrm{T}} M(\dot{x},x,t)u$$

$$\dot{\omega}_2 = K_1(\dot{x},x,t)+\left(\frac{\partial\omega_2}{\partial \dot{x}}\right)^{\mathrm{T}} M(\dot{x},x,t)u \tag{2.1}$$

Here

$$B_1 = \left(\frac{\partial\omega_1}{\partial x}\right)^{\mathrm{T}} A_0^{-1}B_0 + \left[\frac{d}{dt}\left(\frac{\partial\omega_1}{\partial x}\right)^{\mathrm{T}}\right]\dot{x}+\frac{d}{dt}\left(\frac{\partial\omega_1}{\partial x}\right)$$

$$K_1 = \left(\frac{\partial\omega_2}{\partial \dot{x}}\right)^{\mathrm{T}} A_0^{-1}B_0 + \left(\frac{\partial\omega_2}{\partial x}\right)^{\mathrm{T}}\dot{x}+\frac{d\omega_2}{dx}; \quad M = A_0^{-1}M_0$$

Multiplying both sides of Eqs (2.1) by the symmetric positive-definite $k \times k$ and $m \times m$ matrices $A(x, t)$ and $N(x, t)$ with bounded, continuous and continuously differentiable elements in the domain G, we obtain

$$A(x,t)\ddot{\omega}_1 = B(\dot{x},x,t)+Q_1; \quad B = AB_1, \quad Q_1 = A(\partial\omega_1/\partial x)^{\mathrm{T}}Mu$$

$$N(x,t)\dot{\omega}_1 = K(\dot{x},x,t)+Q_2; \quad K = NK_1, \quad Q_2 = N(\partial\omega_2/\partial\dot{x})^{\mathrm{T}}Mu \tag{2.2}$$

In the first equation of (2.2), instead of $\dot{\omega}_1$, we make the substitution

$$y = \dot{\omega}_2 - f(\omega_1,t), \quad f(0,t)=0 \tag{2.3}$$

where $f(\omega_1, t)$ is an arbitrary k-dimensional vector function with bounded and differentiable elements which admit of an infinitely small upper limit.

Multiplying the first equation of (2.2) scalarly by the vector y and the second equation of (2.2) by the vector $\dot{\omega}_2$, and adding, we obtain

$$\frac{1}{2}\frac{d}{dt}(\omega_2^{\mathrm{T}}N\omega_2 + y^{\mathrm{T}}Ay) = y^{\mathrm{T}}\left\{Q_1 + B + A\left(\frac{\partial f}{\partial\omega_1}\right)^{\mathrm{T}} y - A\left[\left(\frac{\partial f}{\partial\omega_1}\right)^{\mathrm{T}} f + \frac{\partial f}{\partial t}\right] + \frac{1}{2}\frac{dA}{dt}y\right\} +$$

$$+\omega_2^{\mathrm{T}}\left(Q_2 + K + \frac{1}{2}\frac{dN}{dt}\omega_2\right) \tag{2.4}$$

If the vectors Q_1 and Q_2 are chosen in the form

$$Q_1 = -Dy - F_1\omega_1 - B - A\left(\frac{\partial f}{\partial\omega_1}\right)^{\mathrm{T}} y + A\left[\left(\frac{\partial f}{\partial\omega_1}\right)^{\mathrm{T}} f + \frac{\partial f}{\partial t}\right] - \frac{1}{2}\frac{dA}{dt}y$$

$$Q_2 = -K - \frac{1}{2}\frac{dN}{dt}\omega_2 - F_2\omega_2 \tag{2.5}$$

we obtain

$$\frac{1}{2}\frac{dV}{dt} = -y^{\mathrm{T}}Dy + \left(f^{\mathrm{T}}F_1 + \omega_1^{\mathrm{T}}\frac{\dot{F}_1}{2}\right)\omega_1 - \omega_2^{\mathrm{T}}F_2\omega_2 \tag{2.6}$$

where D, F_1, F_2 are symmetric, positive-definite matrices and $V = \omega_2^{\mathrm{T}}N\omega_2 + y^{\mathrm{T}}Ay + \omega_1^{\mathrm{T}}F_1\omega_1$ is Lyapunov's function, which is constructed to be positive-definite for all y, ω_1, ω_2, t in the domain G and which admits of an infinitesimal upper limit. Consequently, when the function $(f^{\mathrm{T}}F_1 + \omega_1^{\mathrm{T}}F_1/2)\omega_1$ is negative-definite, the right-hand side of equality (2.6) will be negative-definite with respect to y, ω_1, ω_2 and, in this case, the programmed manifold (1.2) will be asymptotically stable in the domain G. In particular, when $f = -\omega_1$, vectors (2.5) have the form

$$Q_1 = -Dy - F_1\omega_1 - B - A\dot{\omega}_1 - \frac{1}{2}\frac{dA}{dt}y$$

$$Q_2 = -F_2\omega_2 - K - \frac{1}{2}\frac{dN}{dt}\omega_2$$

Here, dA/dt, dN/dt is assumed to be bounded in G in the same way as df/dt.

Note that Q_1 and Q_2 are expressed in terms of the vector u by the equalities presented in (2.2), which can be represented by the single $(k+m)$-dimensional vector equation

$$\Omega u = Q;\quad \Omega = \begin{Vmatrix} A(\partial\omega_1/\partial x)^{\mathrm{T}} A_0^{-1} M_0 \\ N(\partial\omega_2/\partial\dot{x})^{\mathrm{T}} A_0^{-1} M_0 \end{Vmatrix},\quad Q = \begin{Vmatrix} Q_1 \\ Q_2 \end{Vmatrix} \tag{2.7}$$

where Ω is a $((k+m)\times r)$-dimensional matrix and Q is a $(k+m)$-dimensional vector.

The solution of Eq. (2.7) can be represented in the form of the sum of two components [3]: $u_n = \Omega^{\mathrm{T}}\lambda$ and u_τ, where u_τ satisfies the equation $\Omega u_\tau = 0$ and λ is a $(k+m)$-dimensional vector, which is found from the equation $\Omega u_n = Q$ in the form [3]

$$\lambda = (\Omega\Omega^{\mathrm{T}})^{-1}Q$$

Consequently, the component u_n of the vector u is found in the form

$$u_n = \Omega^{\mathrm{T}}(\Omega\Omega^{\mathrm{T}})^{-1}Q \tag{2.8}$$

If the component u_τ of the vector u is equated to zero, the vector u will have the form (2.8) and have a minimum Euclidean norm [3]. In this case, by analogy with constrained mechanical systems, the constraints (1.2) can be regarded as ideal.

3. ESTIMATE OF THE QUALITY OF THE TRANSIENT

Integrating both sides of Eq. (2.6) with respect to time, we obtain

$$\int_{t_0}^{\infty}\left[\omega_2^{\mathrm{T}}F_2\omega_2 + y^{\mathrm{T}}Dy - \left(f^{\mathrm{T}}F_1 + \omega_1^{\mathrm{T}}\frac{\dot{F}_1}{2}\right)\omega_1\right]dt = \frac{1}{2}V_0, \tag{3.1}$$

This equality is an integral criterion of the quality of the transient. Being free to choose the matrices D, F_1, F_2, N, A and the function f, it is possible to give the integrand and Lyapunov's function the required structure with the necessary weight elements.

When the actual numerical value of V_0 is specified, the equation

$$\frac{1}{2}(\omega_{20}^{\mathrm{T}}N\omega_{20} + y_0^{\mathrm{T}}Ay_0 + \omega_{10}^{\mathrm{T}}F_1\omega_{10}) = V_0 \tag{3.2}$$

in the $(2k+m)$-dimensional space $\dot{\omega}_{10}$, ω_{10}, ω_{20} describes an ellipsoid, the surface of which is the geometric locus of points possessing the following property. The integral criterion for the quality of the transient (3.1) holds for motions which have started out from them, and the estimate of the quality of the transient

$$\int_{t_0}^{\infty}\left[\omega_2^{\mathrm{T}}F_2\omega_2 + y^{\mathrm{T}}Dy - \left(f^{\mathrm{T}}F_1 + \omega_1^{\mathrm{T}}\frac{\dot{F}_1}{2}\right)\omega_1\right]dt < \frac{1}{2}V_0$$

holds for all initial values of $\dot{\omega}_{10}$, ω_{10}, ω_{20} within ellipsoid (3.2).

4. CONSTRUCTION OF A SYSTEM WITH BANG-BANG CONTROL

Suppose system (2.2) satisfies the following conditions in the domain G

$$Q_i^{(1)} > \left|G_i^{(1)}\right|, \quad i = 1,2,\ldots,k; \quad Q_j^{(2)} > \left|G_j^{(2)}\right|, \quad j = 1,2,\ldots,m$$

where $G_i^{(1)}$ and $G_j^{(2)}$ are, respectively, the elements of the vectors

$$G^{(1)} = B + A\left(\frac{\partial f}{\partial \omega_1}\right)^{\mathrm{T}} y - A\left[\left(\frac{\partial f}{\partial \omega_1}\right)^{\mathrm{T}} f + \frac{\partial f}{\partial t}\right] + \frac{1}{2}\frac{dA}{dt} y$$

$$G^{(2)} = K + \frac{1}{2}\frac{dN}{dt}\omega_2$$

which occur in expression (2.4), and $G_i^{(1)}$ and $G_j^{(2)}$ are the moduli of the elements of the vectors Q_1 and Q_2 in the case of bang-bang control.

Then, the possibility of bang-bang control

$$Q_{1i} = -Q_i^{(1)} \operatorname{sign} y_i, \quad i = 1,2,\ldots,k$$

$$Q_{2j} = -Q_j^{(2)} \operatorname{sign} \omega_{2j}, \quad j = 1,2,\ldots,m \tag{4.1}$$

follows from (2.4).

Here, the function $V_1 = \omega_2^{\mathrm{T}} N \omega_2 + y^{\mathrm{T}} A y$ and, together with it, the vectors ω_2 and y vanish after a finite time interval [4]. This means that, after this time interval, the phase state of the system is brought into the manifold

$$\dot{\omega}_1 - f(\omega_1, t) = 0, \quad \omega_2 = 0 \tag{4.2}$$

If the function $f(\omega_1, t)$ is chosen in such a way that the solution $\omega_1 = 0$ of the first equation of (4.2) is exponentially stable in G, then the phase state of system (2.1) will contract with time to the programmed manifold (1.2).

It now remains to express the vector u in terms of the quantities (4.1). To do this, the vectors Q_1 and Q_2, occurring in Eq. (2.7), must be replaced by the vectors Q'_1, Q'_2 with the elements

$$Q'_{1i} = -Q_i^{(1)} \operatorname{sign} y_i, \quad i = 1,2,\ldots,k$$
$$Q'_{2j} = -Q_j^{(2)} \operatorname{sign} \omega_{2j}, \quad j = 1,2,\ldots,m$$

respectively.

In this case, Eq. (2.7) takes the form

$$\Omega u = Q'; \quad \begin{Vmatrix} Q'_1 \\ Q'_2 \end{Vmatrix}$$

and the required Eq. (2.8), which has the minimum Euclidean norm, is expressed in the form

$$u = \Omega^{\mathrm{T}}(\Omega\Omega^{\mathrm{T}})^{-1} Q'$$

Note that the expression for the component u_τ of the control u can be found in [5] in the form $u_\tau = [E - \Omega^{\mathrm{T}}(\Omega\Omega^{\mathrm{T}})^{-1}\Omega]W$, where E is the identity matrix and W is an arbitrary vector function.

5. EXAMPLES

Stabilization of the programmed manifold of a manipulator on a moving base. Consider a manipulator, consisting of n rectilinear elements T_ν ($\nu = 1, 2, \ldots, n$), located on a moving base [6]. Each element T_ν rotates relative to the preceding element $T_{\nu-1}$ around a circular cylinder $O_{\nu-1}$. We place the centre of the gripping device at the point O_n of the last element T_n and we specify the plane of the gripping device to be the plane of the unit vectors $\mathbf{e}_p$ and $\mathbf{k}_p$.

We shall assume that rotation of the element T_ν around the hinge $O_{\nu-1}$ is achieved by means of an electric motor, placed with its reducing gear train at this point. The first element of the manipulator is connected to the point O of the base of the manipulator. The position of the body and base of the manipulator relative to a fixed system of coordinates $o\varphi\eta\zeta$ is determined by the law of motion $\mathbf{r}_0(t)$ of the point O and by the three Euler angles.

We impose two requirements in the programme for the motion of the gripping device: the motion of the centre of the gripping device relative to the base must obey the specified law

$$\mathbf{O}_0\mathbf{O}_n = \mathbf{L}(t);\quad \mathbf{L}(t) = \mathbf{L}_1(t) - \mathbf{r}_0(t) \tag{5.1}$$

where $\mathbf{L}_1(t)$, $\mathbf{r}_0(t)$ are specified functions and the plane of the gripping device must be orthogonal to the vector $\mathbf{e}_0$, that is,

$$\mathbf{e}_p^{\mathrm{T}}\mathbf{e}_0 = 0,\quad \mathbf{k}_p^{\mathrm{T}}\mathbf{e}_0 = 0 \tag{5.2}$$

If the vector $\mathbf{O}_0\mathbf{O}_n$ is expressed in terms of the vectors $\mathbf{I}_\nu = \mathbf{O}_{\nu-1}\mathbf{O}_\nu$, then condition (5.1) takes the form

$$\sum_{\nu=1}^{n} \mathbf{l}_\nu - \mathbf{L}(t) = 0 \tag{5.3}$$

The motions of the manipulator are specified by Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = Q(p) - C\dot{x} + M_0 u \tag{5.4}$$

where x is an n-dimensional vector of the generalized coordinates and the angles of rotation $\varphi_\nu(\nu = 1, \ldots, n)$ of its elements, u is the n-dimensional vector of the control signals, $Q(p)$ is the vector of the generalized gravitational forces, T is the kinetic energy of the manipulator

$$T = \frac{1}{2}\dot{x}^{\mathrm{T}}A_0(x,t)\dot{x} + b^{\mathrm{T}}(x,t)\dot{x} + b_0(x,t)$$

$$C = \mathrm{diag}(c_1k_1^2, c_2k_2^2, \ldots, c_nk_n^2);\quad M_0 = \mathrm{diag}(a_1k_1, a_2k_2, \ldots, a_nk_n)$$

c_ν the coefficients of resistance on the shaft of the motors, k_ν are the transmission numbers of the reducing gear trains and a_ν are the coefficients of proportionality between the control moments of the motors and the control signals u_ν.

Equation (5.4) can be represented in the form of (1.1)

$$A_0(x,t)\ddot{x} = B_0(\dot{x},x,t) + M_0 u \tag{5.5}$$

where

$$B_0 = \frac{1}{2}\dot{x}^{T}\frac{\partial A_0}{\partial x}\dot{x} + \left(-\frac{dA_0}{dt} + \frac{\partial b}{\partial x} - C\right)\dot{x} - \frac{db}{dt} + \frac{\partial b_0}{\partial x} + Q(p)$$

Starting out from requirements (5.3) and (5.2), we will represent the programmed manifold (1.2) in the form of a five-dimensional vector

$$\omega(x, t) = 0 \tag{5.6}$$

with elements

$$\omega_i = \mathbf{k}_i^{\mathrm{T}}\left[\sum_{\nu=1}^{n} l_\nu \mathbf{e}_\nu - \mathbf{L}(t)\right],\quad i = 1,2,3;\quad \omega_4 = \mathbf{e}_p^{\mathrm{T}}\mathbf{e}_0,\quad \omega_5 = \mathbf{k}_p^{\mathrm{T}}\mathbf{e}_0 \tag{5.7}$$

where $\mathbf{k}_j$ are the unit vectors for the axes of the system of coordinates $o\varphi\eta\zeta$ and $\mathbf{e}_\nu$ are the unit vectors of the vectors $\mathbf{l}_\nu$.

Differentiating equalities (5.7) twice with respect to t using Poisson's formulae

$$\dot{\mathbf{e}}_\nu = \left(\boldsymbol{\omega}_0 + \sum_{\eta=1}^{\nu} \boldsymbol{\omega}'_\eta\right) \times \mathbf{e}_\nu, \quad \dot{\mathbf{e}}_p = \left(\boldsymbol{\omega}_0 + \sum_{\eta=1}^{n} \boldsymbol{\omega}'_\eta\right) \times \mathbf{e}_p$$

$$\dot{\mathbf{k}}_p = \left(\boldsymbol{\omega}_0 + \sum_{\eta=1}^{n} \boldsymbol{\omega}'_\eta\right) \times \mathbf{k}_p$$

where $\boldsymbol{\omega}'_\eta = \dot{x}_\eta \mathbf{j}_\eta$ and $\mathbf{j}_\eta$ are the unit vectors of $\boldsymbol{\omega}'_\eta$, we obtain Eq. (2.1) in the form [6]

$$\ddot{\omega} = A_1^{\mathrm{T}} A_0^{-1} B_0 + \dot{A}_1^{\mathrm{T}} \dot{x} - \dot{c}_0 + A_1^{\mathrm{T}} A_0^{-1} M u \tag{5.8}$$

Here A_1 is an $n \times 5$ matrix with elements

$$a_{i\nu} = \left(\mathbf{j}_\nu \times \sum_{\eta=1}^{n} \mathbf{l}_\eta\right)^{\mathrm{T}} \mathbf{k}_i, \quad i = 1, 2, 3$$

$$a_{4\nu} = (\mathbf{j}_\nu \times \mathbf{e}_p)^{\mathrm{T}} \mathbf{e}_0, \quad a_{5\nu} = (\mathbf{j}_\nu \times \mathbf{k}_p)^{\mathrm{T}} \mathbf{e}_0$$

c_0 is a vector with elements

$$c_i^0 = \left[\dot{\mathbf{L}}(t) - \sum_{\nu=1}^{n} (\boldsymbol{\omega}_0 \times \mathbf{l}_\nu)\right]^{\mathrm{T}} \mathbf{k}_i, \quad i = 1, 2, 3$$

$$c_4^0 = -(\boldsymbol{\omega}_0 \times \mathbf{e}_p)^{\mathrm{T}} \mathbf{e}_0 - \mathbf{e}_p^{\mathrm{T}} \dot{\mathbf{e}}_0, \quad c_5^0 = -(\boldsymbol{\omega}_0 \times \mathbf{k}_p)^{\mathrm{T}} \mathbf{e}_0 - \mathbf{k}_p^{\mathrm{T}} \dot{\mathbf{e}}_0$$

and $\boldsymbol{\omega}_0$ is the angular velocity of the base..

Multiplying (5.8) by the symmetric, positive-definite 5×5 matrix $A(x, t)$, we obtain

$$A\ddot{\omega} = B(\dot{x}, x, t) + Q \tag{5.9}$$

where

$$B = A(A_1^{\mathrm{T}} A_0 B_0 + \dot{A}_1^{\mathrm{T}} \dot{x} - \dot{c}_0), \quad Q = A A_1^{\mathrm{T}} M u, \quad M = A_0^{-1} M_0$$

Equation (5.9) is the first of the equations of (2.2).

Consequently, Eq. (2.7) for determining the vector u has the form

$$A A_1^{\mathrm{T}} M u = Q \tag{5.10}$$

where

$$Q = -Dy - F\omega - B - A\left(\frac{\partial f}{\partial \omega}\right)^{\mathrm{T}} y + A\left[\left(\frac{\partial f}{\partial \omega}\right)^{\mathrm{T}} f + \left(\frac{\partial f}{\partial t}\right)^{\mathrm{T}} y\right] - \frac{1}{2}\frac{dA}{dt} y$$

D and F are arbitrary symmetric, positive-definite matrices and $f(\omega, t)$ is an arbitrary vector function with bounded and differentiable elements, which is chosen in such a way that the function $(f^{\mathrm{T}} F + \omega^{\mathrm{T}} \dot{F}/2)\omega$ is negative-definite.

The solution of Eq. (5.10) can be represented in the form of (2.8)

$$u = \Omega^{\mathrm{T}} (\Omega \Omega^{\mathrm{T}})^{-1} Q; \quad \Omega = A A_1^{\mathrm{T}} M$$

In this case, the Euclidean norm of the vector of the control u will be a minimum.

When necessary, a bang-bang control of the system can be constructed using the algorithm presented in Section 4.

Note that, in this case, the quality factor of the transient (3.1) has the form

$$\int_{t_0}^{\infty} \left[y^{\mathrm{T}} D y - \left(f^{\mathrm{T}} + \omega^{\mathrm{T}} \frac{\dot{F}}{2} \omega\right)\right] dt = \frac{1}{2} V_0, \quad V_0 = V(t_0)$$

Stabilization of the programmed orientation of a pursuing body. Consider a rigid body rigidly associated with the fixed system of coordinates *cxyz*. We construct the principal vector of the control forces $\tilde{u}_2$ in such a way that the centre of mass of the body *c* moves along the pursuit curve behind the point *o*, which is being pursued, when it moves in an arbitrary way $r_0(t)$ relative to the inertial system of coordinates $o_1x_1y_1z_1$ [7]. Here, the vector of the control moments $\tilde{u}_1$ must be such that one axis of the body, *cz*, for example, tends asymptotically to occupy the direction *co*. Consequently, the programmed orientation of the body in this case can be specified by the expressions

$$\omega_1^i = \mathbf{k}_i \cdot (\mathbf{r}_0 - \mathbf{r}) = 0, \quad \omega_2^i = \mathbf{k}_i \cdot \mathbf{v} = 0, \quad i = 1,2 \tag{5.11}$$

where $\mathbf{k}_1$, $\mathbf{k}_2$ are the unit vectors of the axes *cx* and *cy*, (.) is the sign of a scalar product, $\mathbf{r}$ is a radius vector and $\mathbf{v}$ is the vector of the absolute velocity of the centre of mass of the body.

It is well known that the motion of the centre of mass in the inertial system of coordinates $o_1x_1y_1z_1$ and the rotational motion of the body about the centre of mass in the axes of the fixed system of coordinates *cxyz* are described by the equations

$$I\dot{\boldsymbol{\omega}} = (I\boldsymbol{\omega} \times \boldsymbol{\omega}) + \mathbf{M}_* + \tilde{\mathbf{u}}_1$$

$$m\dot{\mathbf{v}} = \mathbf{f} + \tilde{\mathbf{u}}_2 \tag{5.12}$$

where I is the inertia tensor of the body at the point *c*, $\boldsymbol{\omega}(p, q, r)$ is the instantaneous angular velocity of the body, *p*, *q* and *r* are the projections of $\boldsymbol{\omega}$ onto the *cx*, *cy* and *cz* axes, *m* is the mass of the body, $\mathbf{M}_*$ is the moment of the specified forces with respect to the centre of mass *c*, $\mathbf{f}$ is the principal vector of the specified forces, $\tilde{\mathbf{u}}_1$ is the principal moment of the control forces with respect to *c* and $\tilde{\mathbf{u}}_2$ is the principal vector of the control forces.

We represent Eq. (5.12) in the form

$$\dot{\boldsymbol{\omega}} = I^{-1}[(I\boldsymbol{\omega} \times \boldsymbol{\omega}) + \mathbf{M}_*] + \mathbf{u}_1; \quad \dot{\mathbf{v}} = \mathbf{f}/m + \mathbf{u}_2$$

$$\mathbf{u}_1 = I^{-1}\tilde{\mathbf{u}}_1, \quad \mathbf{u}_2 = \tilde{\mathbf{u}}_1/m \tag{5.13}$$

The problem can now be formulated as follows: it is required to construct analytic expressions for the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in such a way that the manifold (5.11) is a stable programmed manifold of system (5.13).

The solution of the problem is as follows.

Differentiating expression (5.11) with respect to time, we obtain, using Eqs (5.13),

$$\frac{d^2\omega_1^i}{dt^2} = B_1^i + \tilde{Q}_1^i, \quad \frac{d\omega_2^i}{dt} = K_1^i + \tilde{Q}_2^i, \quad i = 1,2 \tag{5.14}$$

where

$$B_1^i = \mathbf{c}_i^{\mathrm{T}} I^{-1}[(I\boldsymbol{\omega} \times \boldsymbol{\omega}) + \mathbf{M}_*] - \mathbf{k}_i^{\mathrm{T}} \frac{1}{m}\mathbf{f} + (\mathbf{r}_0 - \mathbf{r})^{\mathrm{T}}[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{k}_i)] + (\dot{\mathbf{r}}_0 - \mathbf{v})^{\mathrm{T}}(\boldsymbol{\omega} \times \mathbf{k}_i) + \ddot{\mathbf{r}}_0^{\mathrm{T}}\mathbf{k}_i$$

$$K_1^i = \mathbf{k}_i^{\mathrm{T}} \frac{1}{m}\mathbf{f} + \mathbf{v}^{\mathrm{T}}(\boldsymbol{\omega} \times \mathbf{k}_i)$$

$$\mathbf{c}_i = \mathbf{k}_i \times (\mathbf{r}_0 - \mathbf{r})$$

$$\tilde{Q}_1^i = \mathbf{c}_i^{\mathrm{T}}\mathbf{u}_1 - \mathbf{k}_i^{\mathrm{T}}\mathbf{u}_2, \quad \tilde{Q}_2^i = \mathbf{k}_i^{\mathrm{T}}\mathbf{u}_2$$

and $\mathbf{r}$, $\mathbf{v}$, $\mathbf{k}_i$, $\mathbf{c}_i$, $\boldsymbol{\omega}$, $\mathbf{M}_*$, $\mathbf{f}$, $\mathbf{u}_1$, $\mathbf{u}_2$ are column vectors.

The system of equations (5.14) can be represented in the vector form

$$\ddot{\omega}_1 = B_1 + \tilde{Q}_1, \quad \dot{\omega}_2 = K_1 + \tilde{Q}_2 \tag{5.16}$$

where

$$\omega_i = \begin{Vmatrix} \omega_i^1 \\ \omega_i^2 \end{Vmatrix}, \quad B_1 = \begin{Vmatrix} B_1^2 \\ B_1^2 \end{Vmatrix}, \quad \tilde{Q}_1 = \begin{Vmatrix} Q_1^1 \\ Q_1^2 \end{Vmatrix}$$

Multiplying Eqs (5.16) by the sign-definite, symmetric matrices A and N, we obtain

$$A\dot{\omega}_1 = B + Q_1, \quad N\dot{\omega}_2 = K + Q_2 \qquad (5.17)$$

where

$$B = AB_1, \quad K = NK_1, \quad Q_1 = A\tilde{Q}_1, \quad Q_2 = N\tilde{Q}_2 \qquad (5.18)$$

Taking the notation of (5.18) into account, expressions (5.15), which contain the required vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, can be represented in the form

$$\Omega_1\mathbf{u} = A^{-1}Q_1, \quad \Omega_2\mathbf{u} = N^{-1}Q_2 \qquad (5.19)$$

where

$$\Omega_1 = \begin{Vmatrix} c_{11} & c_{12} & c_{13} & -k_{11} & -k_{12} & -k_{13} \\ c_{21} & c_{22} & c_{23} & -k_{21} & -k_{22} & -k_{23} \end{Vmatrix}$$
$$\Omega_2 = \begin{Vmatrix} 0 & 0 & 0 & k_{11} & k_{12} & k_{13} \\ 0 & 0 & 0 & k_{21} & k_{22} & k_{23} \end{Vmatrix} \qquad (5.20)$$

c_{i1}, c_{i2}, c_{i3} are the elements of the vectors $\mathbf{c}_i$, and k_{i1}, k_{i2} and k_{i3} are the elements of the vectors $\mathbf{k}_i$.

In Eq. (5.17), instead of ω_1, we make the substitution

$$y = \dot{\omega}_1 - f(\omega_1, t), \quad f(0,t) = 0 \qquad (5.21)$$

where $f(\omega_1, t)$ is an arbitrary two-dimensional vector function with bounded and differentiable elements, which admit of an infinitesimal upper limit.

Multiplying the first equation of (5.17) scalarly by the vector y and the second equation by the vector ω_2 and adding, we obtain Eq. (2.4).

If the vectors Q_1 and Q_2 are chosen in the form (2.5), we then obtain Eq. (2.6).

Note that Q_1 and Q_2 are expressed in terms of the 6-dimensional vector $\mathbf{u}$ by Eqs (5.19), which can be represented by a single 4-dimensional vector equation of the form (2.8), where Ω is a 4×6 matrix of the form

$$\Omega = \begin{Vmatrix} \Omega_1 \\ \Omega_2 \end{Vmatrix}; \quad Q = \begin{Vmatrix} A^{-1}Q_1 \\ N^{-1}Q_2 \end{Vmatrix}$$

where Ω_1, Ω_2 are the matrices (5.20).

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